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GEOMETRICALLY NON-LINEAR DYNAMIC MODEL OF A ROTATING FLEXIBLE ARM

B. O. AL-BEDOOR AND M. N. HAMDAN

Mechanical Engineering Department, King Fahd University of Petroleum and Minerals, KFUPM Box 841, *Dhahran* 31261, *Saudi Arabia. E-mail*: *bobedoor@kfupm.edu.sa*

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In this paper, a mathematical model for a rotating flexible arm undergoing large planar flexural deformations is developed. The position of a typical material point along the span of the arm is described by using the inertial reference frame via a transformation matrix from the body co-ordinate system which is attached to the root of the rotating arm. The condition of inextensibility is employed to relate the axial and transverse deflections of the material point. The position and velocity vectors obtained, after imposing the inextensibility conditions, are used in the kinetic energy expression while the exact curvature is used in the potential energy. The Lagrangian dynamics in conjunction with the assumed modes method is utilized to derive directly the equivalent temporal equations of motion. The resulting non-linear model is discussed, simulated and the result of simulation are presented and compared to those obtained from the linear theory for different arm parameters.

1. INTRODUCTION

The use of light-weight structural elements in space applications as well as in robotic manipulations under the requirement of precise positioning has increased interest in modelling the flexibility of such structures' and its effect on the overall rigid/flexible body dynamics. Based on such structures inherent lateral flexibility and on the fact of axial rigidity, the problem of the axial displacement due to bending deformations has been identified as a major contributor to what is known as geometric stiffening. Geometric stiffening, due to axial shortening, was shown to have a very important role on the stability of such rotating flexible structures and on their positioning control. It became evident that a model of a rotating long slender flexible arm that takes care of the effect of axial displacement, due to bending deformations, based on physically justified geometrical considerations is desirable. This model should allow large flexible rotation of the arm, adopt the condition of inextensibility and account for flexible–rigid modes coupling.

The effect of rotation on the natural frequencies and mode shapes of a rotating beam was reported earlier by Shilhansi [1] and Prudli [2]. These studies have shown that the rotation speed strengthens the beam and result in higher natural frequencies. Likins [3] reported a study on the mathematical modelling of spinning elastic bodies. In the same direction, Kaza and Kavternik [4] reported results of a study on the non-linear flap-lag-axial equations of a rotating beam. They addressed the problem of axial rigidity and the shortening due to transverse deflection. Reference [4] summarized the four methods for accounting the beam axial rigidity as the approach for considering the Green axial strain for the axial deformation, using the inertial effect, artificially, in the kinetic energy or the potential energy or both. Stephen and Wang [5] studied the effect of uniform high-speed rotation on the stretching and bending of a rotating beam. They accounted for the beam rotation in terms of tensile force that produced axial stress on the beam. They accounted for the beam rotation in terms of tensile force that produced axial stress on the beam. In the aforementioned studies, the effect of rotation was taken as kinematic variable in the form of angular velocity and angular acceleration to be given to the elastic equations which in turn are solved for the natural frequencies and mode shapes. Kane *et al.* [6] studied the dynamic behaviour of a cantilever beam that is attached to rigid base which is performing specified motion of rotation and translation. In their work the elastic degrees of freedom included beam axial extension, bending in two planes, torsion, shear displacement and wrapping. The model is a general three-dimensional elastic beam model; however, the two-way coupling between the rigid-body motion and elastic deflections was not accounted for as only specified rigid-body motions were considered.

The multibody dynamic approach, in which the rigid motion and flexible deformation are modelled in their coupled form, has attracted many researchers. Baruh and Tadikonda [7] reported some issues on the dynamics and control of flexible robot manipulators. They addressed the problem of axial shortening due to bending deformations by considering the shortening in their kinetic energy expression. Their results have shown that the flexibility of the rotating arm has changed the desired final rigid-body position. Tadikonda and Chang [8] reported the effect of end load, due to chain connections on the geometric stiffening. Yigit *et al.* [9] studied the dynamics of a radially rotating beam with impact. They modelled the rigid-body motion and the beam elastic co-ordinates using a partial differential equation and Galerkin's method of approximation. The effect of beam axial shortening due to bending deformation and the resulting beam stiffening was considered in their equations. However, the model showed linear inertial coupling between the beam rigid-body rotation and its elastic deflections and the effects of shortening appeared as function of square of beam rigid-body rotating speed in the stiffness term. Pan *et al.* [10] reported a dynamic model and simulation results of a flexible robot arm with prismatic joint. They accounted for the effect of axial shortening using a virtual work term added to the elastic potential energy. El-Absy and Shabana [11] studied the geometric stiffness for a rotating beam using different approaches. They introduced the effect of longitudinal deformation due to bending, in the equations of motion, using the principle of virtual work. Recently, Al-Bedoor [12] studied the effects of shaft torsional flexibility on the dynamics of rotating blades. The effect of axial shortening was accounted for by using the virtual work in the form of added potential energy due to the centrifugal forces. Numerical simulations have shown that the flexibility and the stiffening effect contribute to the rigid-body inertia by quadratic terms.

To this end, one can summarize that the available dynamic models for the rotating beam that included the effect of axial shortening can be classified into two groups. One group was concerned with the effect of uniform spinning speed on the natural frequencies and mode shapes. The second group is concerned with the overall rigid-body and flexible system motions. With regard to accounting for the axial shortening due to bending deformations, the published work can also be classified into two main categories: one category that considered the effect in the form of axial potential energy virtual work and the other that considered the axial shortening, which is found from the binomial expansion, in the kinetic energy. A consistent approach that accounts for the beam axial shortening and its associated non-linear effects that develop by utilizing the beam in-extensibility condition [13, 14] as a constraint for relating axial and transverse elastic position of a typical material point is very important.

This work presents a mathematical model and simulation results for a rotating flexible slender arm. The multibody dynamic approach is followed in developing the model through attaching body co-ordinate system to the hub at the root of the arm. The position vector of a typical material point is used in deriving the kinetic energy expression which includes the rigid-body motion, the arm transverse deflection and its associated axial shortening. The geometrically exact curvature is employed in expressing the arm elastic potential energy. The system Lagrangian in conjunction with the assumed modes method and after imposing the beam inextensibility constraint is used to develop the non-linear system of equations of motion that constitute the rigid-body motion and the arm transverse deflection modal co-ordinates. The obtained non-linear and coupled model is discussed, simulated and the results are presented and compared with those obtained from the linear theory for different arm/hub and motion parameters.

2. THE ELASTODYNAMIC MODEL

2.1. SYSTEM DESCRIPTION AND ASSUMPTIONS

Figure 1 shows a schematic of the beam under consideration. XYZ denotes the inertial reference frame that is fixed in space with the origin O at the centre of the hub, while the xyz is the system of orthogonal axes rotating with the hub with its origin fixed to the root of the beam and the x-axis is oriented along the neutral axis of the beam in the undeflected configuration. The hub is assumed to be rigid with radius R_H and rotating about the global z-axis. The beam is assumed to be initially straight, cantilevered at the based having uniform cross-section A, flexural rigidity EI, constant length l and mass per unit length ρ. The thickness of the beam is assumed to be small compared to the beam length so that the effects of shear deformations and rotary inertia can be neglected. The beam motion is assumed to be confined to the x–y plane (i.e., only in-plane flexural motion is allowed). Furthermore, it is assumed that the peak amplitude in this planar flexural vibrations may reach relatively large values (can be of the order of the beam length for the lower modes), but the slope of the elastica may not have tangents perpendicular to the neutral axis; also the beam is assumed to be conservative. The effect of shortening due to beam transverse deformation, determined by using the inextensibility condition [13, 14], and its time derivative is used to eliminate the dependence of the beam Lagrangian on the axial displacement and the axial velocity. In the following sections, the governing temporal equations of motion are formulated via a combined Lagrangian-assumed mode method.

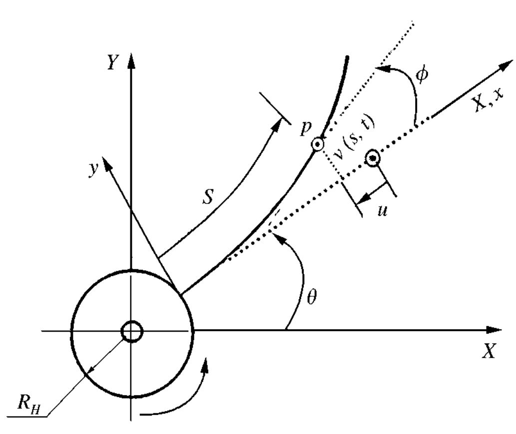

Figure 1. Schematic of the rotating inextensible flexible arm.

2.2. THE KINETIC ENERGY EXPRESSION

To develop the kinetic energy expression for the rotating arm–hub system, the deformed configuration of the arm, shown in Figure 1, is used. The global position vector of a material point P, on the arm, can be written as

$$R_P = R_H + [A(\theta)]r_P, \tag{1}$$

where r_P is the position vector of point P in the hub co-ordinate system xy, $[A(\theta)]$ is the rotational transformation matrix from the hub co-ordinate system to the inertial reference frame, XY, and $\mathbf{R}_H$ is the position vector of the origin of the hub co-ordinate system xy in the inertial reference frame, i.e.,

$$R_H = R_H \cos\theta I + R_H \ \sin\theta J. \tag{2}$$

The position vector of the material point P in the xy co-ordinate system can be written in the form

$$r_P = (s - u(s, t))i + v(s, t)j, \tag{3}$$

where s is the undeflected position, $u(s, t)$ is the axial shortening due to bending deformation and $v(s, t)$ is the transverse deflection of the material point p measured with respect to the hub co-ordinate system, xy, which has the unit vectors i and j. The rotational transformation matrix $[A(\theta)]$ can be represented as

$$[A(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{4}$$

where θ represents the arm rigid-body rotation.

The velocity vector of the material point P in the inertial reference frame can be obtained by differentiating equation (1) as follows:

$$\dot{R}_P = \dot{R}_H + [A(\theta)]\dot{r}_P + \dot{\theta}[A_\theta(\theta)]r_P, \tag{5}$$

where $[A_\theta]$ is the derivative $[\mathrm{d}A/\mathrm{d}\theta]$.

Upon substituting for R_H, $[A_\theta]$, r_P and $\dot{r}_P$ into equation (5), the velocity vector of the material point P in the inertial reference frame can be represented in the form

$$\dot{R}_P = \begin{Bmatrix} -\alpha\sin\theta + \beta\cos\theta \\ \beta\cos\theta + \alpha\sin\theta \end{Bmatrix}, \tag{6}$$

where

$$\alpha = \dot{\theta}(\mathrm{R}_H + s - u) + \dot{v}, \quad \beta = -\dot{u}(s, t) - \dot{\theta}v. \tag{7}$$

The kinetic energy of the beam can be found from

$$U_B = \tfrac{1}{2}\int_0^l \rho\dot{R}_P^{\mathrm{T}'} \cdot \dot{R}_P \,\mathrm{d}s, \tag{8}$$

where ρ is the beam mass per unit length and l is the beam length.

Substituting equation (6) into equation (8) yields the beam kinetic energy expression in the form

$$U_B = \tfrac{1}{2}\rho \int_0^l \begin{bmatrix} R_H^2\dot\theta^2 + (s-u)^2\dot\theta^2 + \dot v^2 + 2R_H(s-u)\dot\theta^2 + 2R_H\dot v\dot\theta \\ + 2(s-u)\dot v\dot\theta + \dot u^2 + v^2\dot\theta^2 + 2v\dot u\dot\theta \end{bmatrix} \mathrm{d}s. \tag{9}$$

The kinetic energy of the hub which is assumed to be a rigid uniform disk with radius R_H and mass m_H and rotating at angular velocity $\dot\theta$, can be written in the form

$$U_H = \tfrac{1}{4} m_H R_H^2 \dot\theta^2. \tag{10}$$

Now, the total kinetic energy expression of the system can be written as

$$U = U_H + U_B. \tag{11}$$

2.2. POTENTIAL ENERGY EXPRESSION

The system potential energy is constituted of the beam elastic strain energy. The arm is assumed to be rotating in the horizontal plane, which results in no gravitational potential energy. The elastic beam strain energy with flexural rigidity $EI(x)$ is given by

$$V_B = \tfrac{1}{2}\int_0^l EI(s)K^2\,\mathrm{d}s, \tag{12}$$

where K is the curvature of the beam centreline at point s, which will be evaluated in the following sections based on the inextensibility condition.

2.3. THE INEXTENSIBILITY CONDITION

For the present two-dimensional beam problem (see Figure 2), the inextensibility condition dictates that total axial shortening $u(s, t)$ is given by [14],

$$\lambda u(\xi, t) = \xi - \int_0^\xi \cos\phi(\eta, t)\,\mathrm{d}\eta, \tag{13}$$

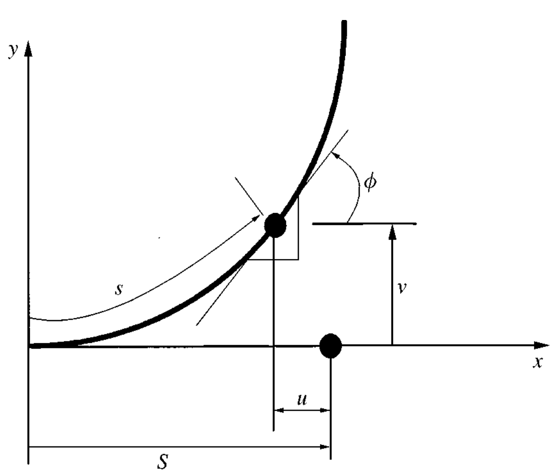

Figure 2. The deformed beam configuration.

where $\xi = s/l$ and $\lambda = 1/l$. Upon noting that $\cos\phi = \sqrt{1-\sin^2\phi}$, $\sin\phi = \mathrm{d}v/\mathrm{d}s$ and expanding the term $[1-(\lambda v'^2)]^{1/2}$ in a power series, assuming that $(\lambda v')^2 \ll 1$, and retaining the terms up to the fourth order, the axial position of the material point can be represented as

$$u = \tfrac{1}{2}\int_0^\xi [\lambda v'^2 + \tfrac{1}{4}\lambda^3 v'^4]\,\mathrm{d}\eta, \tag{14}$$

where the prime is the derivative with respect to the dimensionless length, ξ. Differentiating equation (14) with respect to time yields

$$u = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^\xi [\lambda v'^2 + \tfrac{1}{4}\lambda^3 v'^4]\,\mathrm{d}\eta\right]. \tag{15}$$

In order to express the exact curvature in terms of the transverse deflection v only, the analysis presented in reference [14] is adopted. Accordingly, one notes that the curvature is

$$K = \phi', \tag{16}$$

where

$$\sin\phi = \lambda v'. \tag{17}$$

Differentiating equation (17) and noting, as before, that $\cos\phi = \sqrt{1-\sin^2\phi}$, $\sin\phi = \mathrm{d}v/\mathrm{d}s$ and expanding the term $[1-(\lambda v'^2)]^{1/2}$ in a power series, assuming that $(\lambda v')^2 \ll 1$, and retaining the terms up to the fourth order leads to

$$K^2 = \lambda^4 v''^2 + \lambda^6 v'^2 v''^2. \tag{18}$$

Upon substituting equation (14) for the axial position and its time derivative, equation (15) and the curvature equation (18) into the kinetic and potential and energy expressions, the Lagrangian of the system is obtained as

$$L = \frac{m_B}{2}\int_0^1 \left\{\begin{aligned} &\left(R_H^2 + \frac{I_H}{\rho l}\right)\dot\theta^2 + \frac{\xi^2}{\lambda^2}\dot\theta^2 - \xi\left[\int_0^\xi (v'^2 + \tfrac{1}{4}\lambda^2 v'^4)\,\mathrm{d}\eta\right]\dot\theta^2 \\ &+ \tfrac{1}{4}\left[\int_0^\xi \lambda v'^2\,\mathrm{d}\eta\right]^2\dot\theta^2 + \dot v^2 + 2R_H\frac{\xi}{\lambda}\dot\theta^2 - R_H\left[\int_0^\xi (\lambda v'^2 + \tfrac{1}{4}\lambda^3 v'^4)\,\mathrm{d}\eta\right]\dot\theta^2 \\ &+ 2\left(R_H + \frac{\xi}{\lambda}\right)\dot v\dot\theta - \left[\int_0^\xi (\lambda v'^2 + \tfrac{1}{4}\lambda^3 v'^4)\,\mathrm{d}\eta\right]\dot v\dot\theta + v^2\dot\theta^2 + \tfrac{1}{4}\left[\left(\int_0^\xi \lambda v'^2\,\mathrm{d}\eta\right)^{\boldsymbol{\cdot}}\right]^2 \\ &+ \left[\int_0^\xi \left(\lambda v'^2 + \frac{\lambda^3 v'^4}{4}\right)\mathrm{d}\eta\right]^* v\dot\theta - \beta^2[v''^2 + (\lambda v' v'')^2] \end{aligned}\right\}\mathrm{d}\xi, \tag{19}$$

where m_B is the mass of the beam, $I_H = \frac{1}{2}m_H R_H^2$ is the moment of inertia of the rigid hub, and $\beta^2 = EI\lambda^4/\rho$ is a dimensionless frequency parameter of the beam.

In addition to the Lagrangian, the virtual work done by the external torque applied at the hub can be represented in the form

$$\delta W = T\delta\theta. \tag{20}$$

2.4. THE ASSUMED MODES METHOD (AMM)

The assumed modes method is used in discretizing the beam elastic deformation, $v(s, t)$ used in the Lagrangian expression relative to the hub co-ordinate system, as follows:

$$v(s,t) = \sum_{i=1}^{N} \phi_i(s) q_i(t), \tag{21}$$

where N is the number of modes, q_i is the vector of modal co-ordinates, which is time dependent, and ϕ_i is the vector of the assumed modes.

Upon substituting the assumed modes approximation (AMM) for the beam deformation, equation (21), the Lagrangian expression becomes

$$L = \frac{m_B l^2}{2}\begin{bmatrix} C_0\dot{\theta}^2 + \beta_1 \dot{q}_i^2 - \beta^2\beta_7 q_i^2 + \beta_2\dot{\theta}^2 q_i^2 + \beta_3\dot{\theta}^2 q_i^4 \\ + \beta_4\dot{\theta}\,\dot{q}_i + \beta_5 q_i^2\dot{q}_i\dot{\theta} + \beta_6 q_i^2\dot{q}_i^2 - \beta^2\beta_8 q_i^4 \end{bmatrix}, \tag{22}$$

where $C_0 = C^2(1 + \frac{1}{2}\mu) + (C/2) + (\frac{1}{3})$ is a dimensionless inertia coefficient with $C = R_H/l$ as the ratio of the hub radius to the beam length and $\mu = m_H/m_B$ as the mass ratio of the hub to the beam and the coefficients β_i are as follows:

$$\beta_1 = \int_0^1 \phi^2\,\mathrm{d}\xi = 1, \quad \beta_2 = \int_0^1 \phi^2\,\mathrm{d}\xi - \int_0^1 \xi\left[\left[\int_0^{\xi} \phi'^2\,\mathrm{d}\eta\right)\mathrm{d}\xi - C\int_0^1\left(\int_0^{\xi} \phi'^2\,\mathrm{d}\eta\right)\mathrm{d}\xi,$$

$$\beta_3 = \tfrac{1}{4}\left[\int_0^1\left[\int_0^{\xi}\phi'^2\,\mathrm{d}\eta\right]^2\right]\mathrm{d}\xi - \int_0^1 \xi\left(\int_0^{\xi}\phi'^4\,\mathrm{d}\eta\right)\mathrm{d}\xi - C\int_0^1\left(\int_0^{\xi}\phi'^4\,\mathrm{d}\eta\right)\mathrm{d}\xi,$$

$$\beta_4 = 2\left[\int_0^1 (C+\xi)\phi\,\mathrm{d}\xi\right], \quad \beta_5 = \int_0^1\left(\int_0^{\xi}\phi'^2\,\mathrm{d}\eta\right)\phi\,\mathrm{d}\xi, \quad \beta_6 = \int_0^1\left(\int_0^{\xi}\phi'^2\,\mathrm{d}\eta\right)^2\mathrm{d}\xi,$$

$$\beta_7 = \int_0^1 \phi''^2\,\mathrm{d}\xi, \quad \beta_8 = \int_0^1 (\phi'\,\phi'')^2\,\mathrm{d}\xi. \tag{23}$$

2.5. THE EQUATIONS OF MOTION

By applying the Euler–Lagrange equation to the system Lagrangian equation (22), the system equations of motion are obtained as

$$[C_0 + \beta_2 q_i^2]\ddot{\theta} + \frac{\beta_4}{2}\left[1 + \frac{\beta_5}{\beta_4}q_i^2\right]\ddot{q}_i + 2\beta_2 q_i\dot{q}_i\dot{\theta} + \beta_5 q_i\dot{q}_i^2 = \frac{T}{m_B l^2}, \tag{24}$$

$$(1 + \beta_6 q_i^2)\ddot{q}_i + \omega^2\left[1 - \left(\frac{\beta_2}{\omega^2} + \frac{2\beta_3}{\omega^2}q_i^2\right)\dot{\theta}^2 + \beta_6\dot{q}_i^2\right]q_i + \frac{\beta_4}{2}\left(1 + \frac{\beta_5}{\beta_4}q_i^2\right)\ddot{\theta} + \frac{2\omega^2\beta_8}{\beta_7}q_i^3 = 0, \tag{25}$$

where $i = 1, n =$ is the ith elastic mode, $\omega^2 = \beta_7 \beta^2$ is the linear natural frequency of the non-rotating beam.

Equations (24) and (25) represent the equations of motion of an inextensible beam which is rotating around its hub centre. The system is a two-coupled non-linear differential equations for the system degrees of freedom θ as the rigid-body rotation and q_i as the beam ith modal degree of freedom. In these equations the terms $\dot{q}^2 q$ and $q^2 \ddot{q}$ are inertial non-linearities due to the kinetic energy of the axial motion which arise as a result of using the inextensibility condition. In equation (25) the cubic term, q^3, is a static hardening non-linearity due to the potential energy and arise as a result of using non-linear curvature. These non-linear terms have appeared in the present model due to adopting the inextensibility condition and did not appear in the reported models that used the artificial virtual work to account for the axial shortening: references [7, 9, 11, 12]. The other non-linear terms in these equations are the same as one obtained when using the linear beam, i.e., linear curvature and assuming uncoupled axial and transverse beam motion. In the next section, the system of equations is simulated numerically and the results for the linear and non-linear approaches for accounting for the axial shortening are compared and discussed.

3. NUMERICAL SIMULATION

The computational process started with evaluating the coefficients, equations (23), for the cantilever beam mode shapes. The Gauss-quadrature 16-point integration scheme was used. The system of non-linear second order equations, equations (24) and (25), is simulated by using a multistep variable order prediction–corrector algorithm. The dimension and material properties of the arm–hub system are given in Table 1.

The inverse dynamic procedure is used to design and open loop positioning torque which accounts for the rigid-body inertia that is known. The torque profile that is employed to rotate the system, with 1 m long flexible arm, an angle of $\pi/6$ in 2 s is shown in Figure 3. The resulting arm angular position is shown in Figure 4, wherein the angular position of the linear under the effect of neglecting inertia coupling between the rigid-rotation and the flexible arm motion is compared to the position when the linear and the non-linear models are simulated. It is shown that the linear and non-linear models have given deviation from the target angular position, which is obtained by the rigid-body model. This deviation can be referred mainly to the inertia coupling between the rigid body and the elastic deflection and not to the non-linearity that exists due to the geometrical shortening. The hub angular velocity is shown in Figure 5 for the rigid body, linear and non-linear models respectively. The elastic linear and non-linear models show that the system has reached an angular velocity greater than that of the equivalent uncoupled model. This indicates that some

TABLE 1

Arm-hub data

Property	Value
Arm length l	1·0 m
Arm mass per unit length, ρ	4·015 kg/m
Arm flexural rigidity, EI	756·0 N/m^2
Hub radius, R_D	0·2 m
Basic hub mass m_H	50 kg

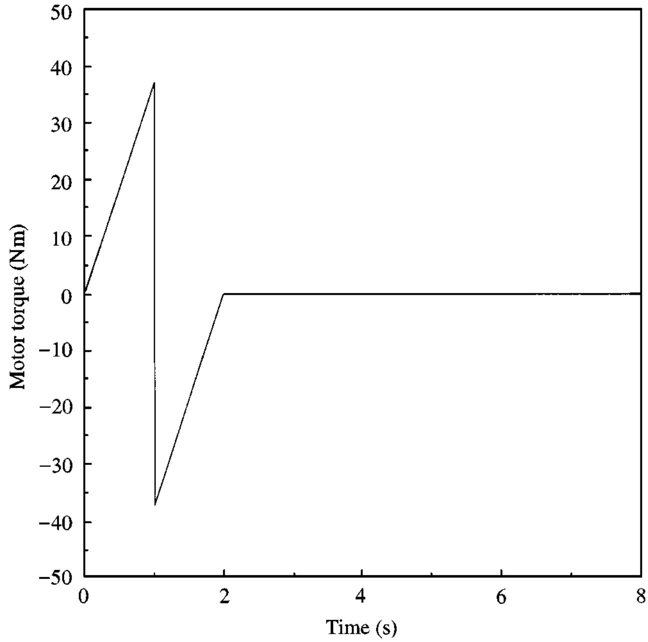

Figure 3. Motor torque to rotate the 1 m arm, an angle $\pi/6$ in 2 s.

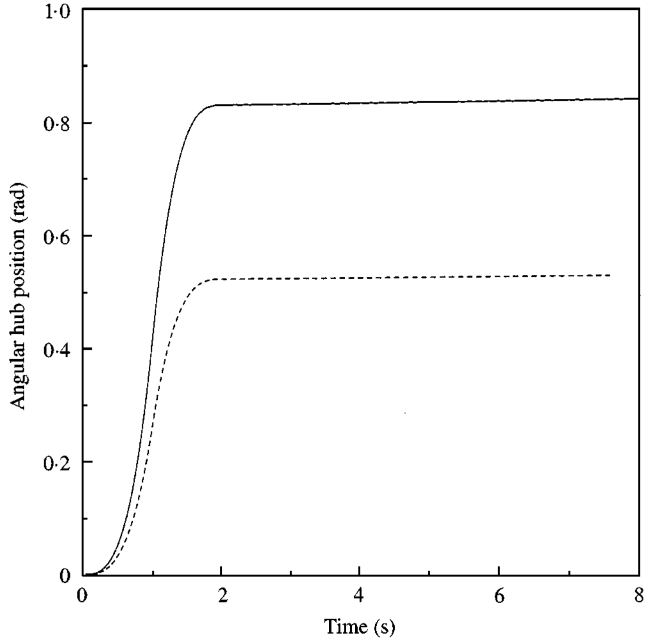

Figure 4. Hub angular position: (– – – –), rigid body; (– · – · –), flexible with linear model; and (——), flexible with non-linear model.

energy, which was originally fed into the elastic mode, is directed to the rigid-body motion through the inertia coupling. Figure 6 shows the tip deflection of the arm measured with respect to the hub co-ordinate system. The tip deflects when the inertia coupling terms are higher than those obtained from the linear and the non-linear models. This shows that some

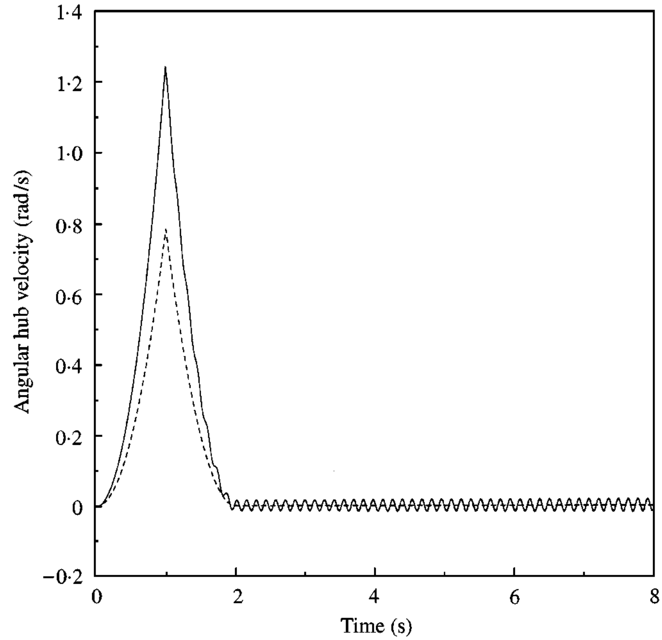

Figure 5. Hub angular velocity: (– – – –), rigid body; (–·–·–), flexible with linear model; and (——), flexible with non-linear model.

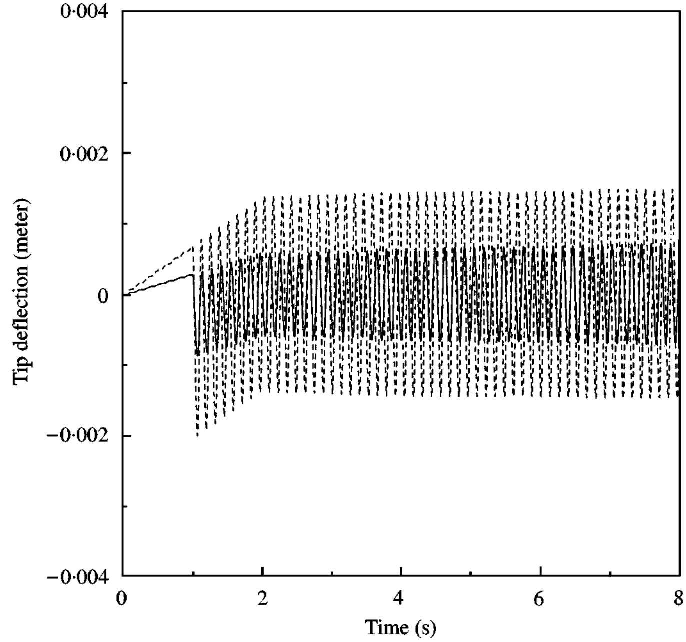

Figure 6. Tip deflection: (– – – –), rigid-body-flexible coupling ignored; (–·–·–), coupled system with linear model; and (——), coupled system with non-linear model.

of the energy is transferred from elastic mode to the rigid-body mode and causes the system to move further from its design point. To further explore the effect of beam length, the beam with a length of 3 m, while all other parameters remain the same, is simulated. The hub

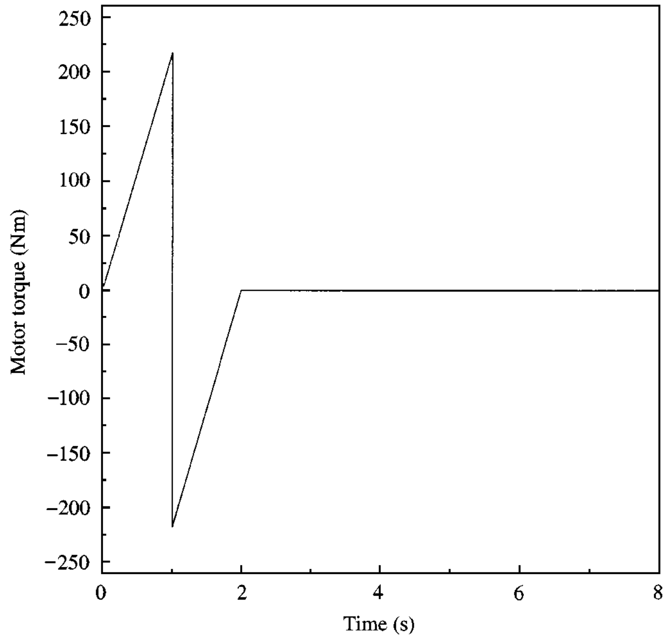

Figure 7. Motor Torque to rotate the 3 m arm, an angle $\pi/6$ in 2 s.

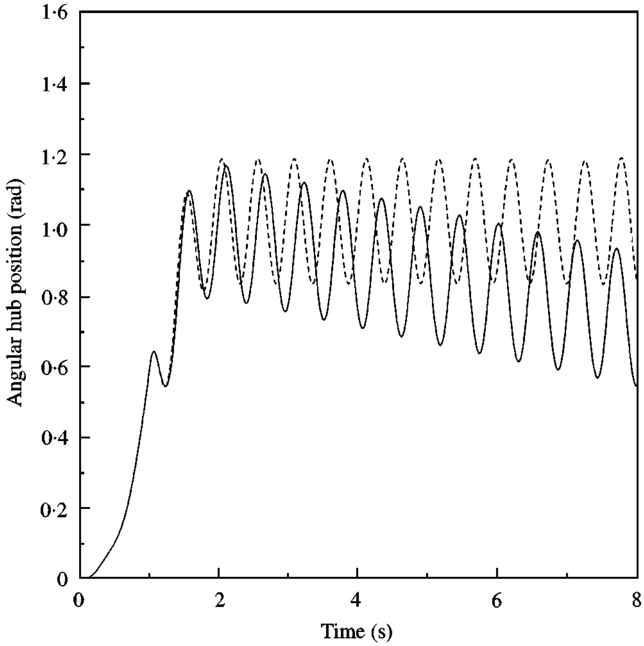

Figure 8. Hub angular position when the arm is 3 m long: (– – – –), linear model; and (——), non-linear model.

torque designed to rotate the 3 m beam–hub system to an angular position of $\pi/6$ in 2 s is shown in Figure 7. The angular position of the system using the linear and non-linear models is shown in Figure 8. It is shown that the final position is oscillatory for both the linear and the non-linear models. The frequency of oscillations is affected by the beam

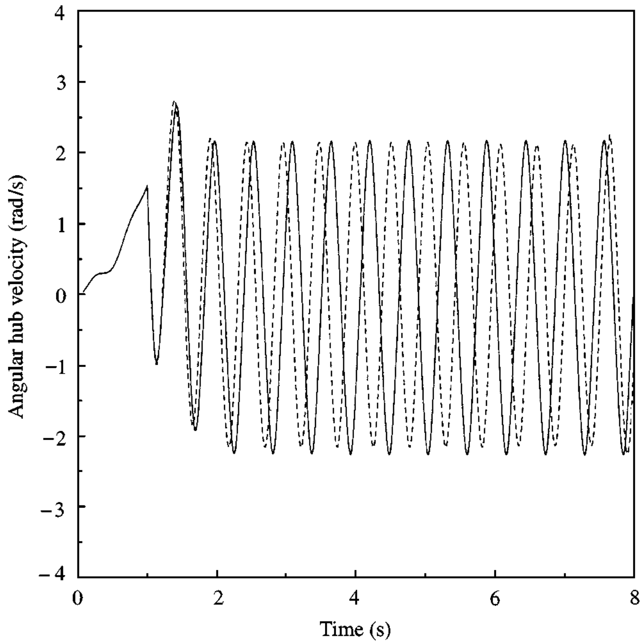

Figure 9. Hub angular velocity when the arm is 3 m long: (– – – –), linear model; and (——), non-linear model.

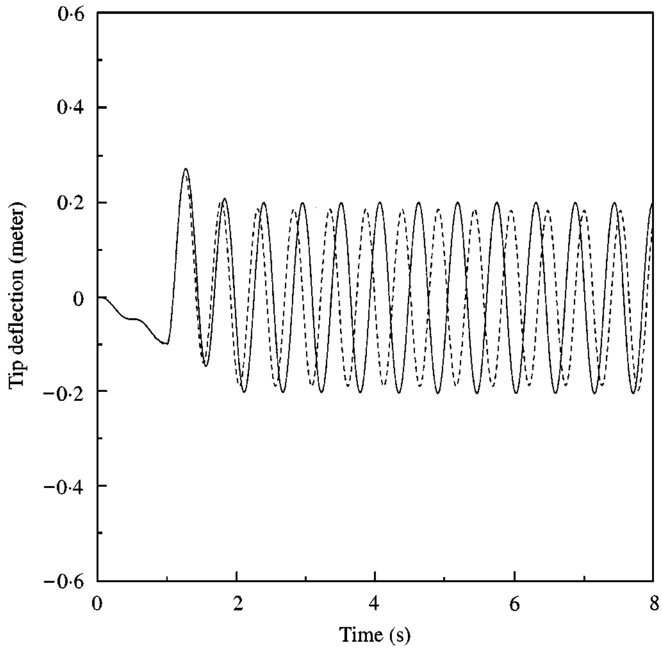

Figure 10. Tip deflection of the 3 m long arm: (– – – –), linear model; and (——), non-linear model.

natural frequency, but tuned due to the non-linearity and resulting in the frequency amplitude relation. Moreover, the non-linear system has described that the end position is moving back from that of the linear system position but towards the rigid-body position response but after some delay. The hub angular velocity, shown in Figure 9, shows that both the linear and the non-linear models angular velocity exhibit oscillations, which indicates that the rigid-body system energy is affected markedly by the elastic deflections. The beam tip deflections obtained from the linear and the non-linear models are shown in Figure 10. The linear model has shown slightly less deflection and higher frequency.

4. CONCLUSIONS

A non-linear dynamic model for a rotating flexible arm is developed in this study. The position and velocity vectors of a deformed material point were described using the inextensible beam theory, which was also utilized in the exact curvature. The equations of motion were derived by using Lagrangian dynamics in conjunction with the assumed modes method for discretizing the beam elastic deflection. Due to the geometrically exact nature of the described position and velocity vectors as well as the exact curvature, the developed equations are suitable for large deflection of the rotating arm, in contrast to the previously reported models. The equations obtained are coupled and non-linear ordinary differential ones that captured the effect of geometrical stiffening without imposing any conditions other than the inextensibility geometrical constraint. The coupled equations of motion showed more non-linear inertial stiffening and softening terms that would not appear without adopting the geometrical inextensibility constraint. The model is simulated and the results are compared with those obtained by using the linear theory and the simulations of the model, which ignores the inertia coupling. The numerical results showed that the added inertia effect comes through the inertia coupling terms and not due to the stiffening effect as was previously reported. The non-linear model simulations showed more realistic results in the positioning accuracy than the linear model results that was obtained due to the consistency in developing the model, in particular when accounting for the axial shortening through imposing the inextensibility condition. More studies on the rotating flexible arm dynamics, control and stability using the reported model are recommended.

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